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Technical Note

"Integer" and "fractional" solutions of Fourier's problem of a ring heated by moving δ-source of energy

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Abstract

The new exact solutions of the classical Fourier's problem of heat conduction in a thin ring with a moving periodical δ-source of energy are obtained. Such kind of solutions of heat conduction equation is not yet studied in the literature. It is shown that they are typical for the heating of rotating hollow bodies and can be used for quantitative description of heat transfer through the moving caterpillar track. Under some conditions the "integer" solution of heat conduction equation can be approximated with a high accuracy by means of a strict succession of positive whole numbers.

1. Introduction

A moving caterpillar track is used widely in engineering. Our strictly practical interest in this field was connected with the investigation of heat transfer through the moving caterpillar track which served as an element of the experimental technique for fabrication of thin solid films. In a first approximation this element can be considered as a system of tubes rolled between two planes heated by different sources of energy. The investigation shows that even the simplest variant of the problem (heating of a single thin-walled tube trundled on a plane with a constant surface density of a heat flow q) has specific stepped solutions that were not yet met in other problems of heat conduction and diffusion. The goal of this note is to demonstrate the main features of these solutions and, more generally, give the possibility of quantitative description of heat transfer in such kind of systems.

2. Model

We suppose the length of a tube, radius of a middle wall surface and the thickness of a tube wall are such that the inequality $l \gg R \gg \Delta r$ is valid, so we can neglect boundary effects near tube butt-ends and the heat flow in radial and axial directions. The motion of the tube can be described by the constant linear speed of a tube axis u and the angular speed of its rotation ω which are connected by the equation $u = \omega R$. By passing to the rotating polar system of coordinates the mathematical model for the description of a temperature field in a cross-section of a tube can be obtained in the following form:

$$\frac{\partial T}{\partial t} = \frac{a}{R^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{q}{c\rho\Delta r}\delta_{2\pi}(\varphi - \omega t), \tag{1}$$

$$T(\varphi, t)|_{t=0} = 0, \tag{2}$$

$$T(\varphi, t)|_{\varphi=-\pi} = T(\varphi, t)|_{\varphi=+\pi}, \tag{3}$$

$$\left.\frac{\partial T(\varphi, t)}{\partial \varphi}\right|_{\varphi=-\pi} = \left.\frac{\partial T(\varphi, t)}{\partial \varphi}\right|_{\varphi=+\pi}, \tag{4}$$

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Nomenclature

a	thermal diffusivity
c	heat capacity at constant pressure
l	length
q	thermal power of a rotating δ-source of energy
R	radius of a middle wall surface
T	temperature
t	time
u	linear speed
α	heat transfer coefficient
Δr	thickness of a tube wall
φ	angular coordinate
ρ	density of a material
ω	angular speed of rotation

2π-periodical Dirac δ-function can be written as a sum of usual δ-functions:

$$\delta_{2\pi}(\varphi-\omega t)=\sum_{k=0}^{\infty}\delta(\varphi+2\pi k+\pi-\omega t). \tag{5}$$

The case of a ring heated by a fixed concentrated source of energy was considered by Fourier [1]. Carslaw and Jaeger note that historically it was the first problem of the analytical theory of heat conduction in solids (Fourier's problem of a ring [2, Chapter 4.15]). The case of a periodical circular movement of δ-source along a ring is not yet studied in the literature.

Turning to the dimensionless variables $\tau=\omega t$, $v=c\rho\Delta r\omega T/q$ and using the Laplace transform

$$y=\int_0^{\infty}\exp(-p\tau)v(\varphi,\tau)\,\mathrm{d}\tau,$$

one can obtain the ordinary differential equation:

$$\frac{\mathrm{d}^2y}{\mathrm{d}\varphi^2}+h^2y+\frac{1}{B}\frac{\exp(-p(\varphi+\pi))}{1-\exp(-2\pi p)}=0, \tag{6}$$

$$y(\varphi)|_{\varphi=-\pi}=y(\varphi)|_{\varphi=+\pi},\ y'(\varphi)|_{\varphi=-\pi}=y'(\varphi)|_{\varphi=+\pi}, \tag{7}$$

where $B=a/\omega R^2$, $h^2=-p/B$.

The exact solution of this equation can be written as:

$$y=-\frac{1}{2B(p^2+h^2)}\left(\frac{\sin(h\varphi)}{\sin(h\pi)}+\frac{p\cos(h\varphi)}{h\sin(h\pi)}+\frac{2\exp(-p(\varphi+\pi))}{1-\exp(-2\pi p)}\right). \tag{8}$$

All terms of this solution have simple poles of order 1 at $p=1/B$ and poles of order 2 at $p=0$. The first and second terms have simple poles at $p=-Bk^2$, where $k=1,2,\ldots$. The residues for these two terms can be found by the standard way and lead to originals of the usual "diffusional" type. In order to find the third term original it is convenient to use the tables of Laplace transform [3] for: $1/(p-1/B)$ and $1/(p(1-\exp(-2\pi p)))$. Using then the convolution and translation theorems we obtain the final solution of the problem (1)–(4) in originals:

$$\frac{c\rho\Delta r\omega T}{q}=\frac{\varphi-B}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k\exp(-Bk^2\tau)}{k(B^2k^2+1)}\times(\sin(k\varphi)-Bk\cos(k\varphi))+\left[\frac{\tau-\varphi-\pi}{2\pi}\right]+1+\frac{\exp\left(-\frac{2\pi}{B}\left(1-\left\{\frac{\tau-\varphi-\pi}{2\pi}\right\}\right)\right)}{1-\exp(-2\pi/B)}. \tag{9}$$

Here and elsewhere $[z]$ mean "the whole part of a number z" – the greatest integer not exceeding the number z, braces $\{z\}$ mean "the fractional part of the number z" and $z=[z]+\{z\}$. The term in the form of infinite series has the usual diffusional type, its limit is zero if $\omega t\to\infty$ and the temperature distribution in a ring will be determined by two next discrete terms having the continuous sum. Such kind of terms is not given in the list of known types of exact solutions of linear parabolic equation of heat conduction [2, Chapter 2.1]. The exponential term with the periodical "fractional" index (the "periodical" exponent) describes the temperature variation of a ring point with the coordinate φ during the 2π period of time from one contact with the δ-source to another. The integer term numbers contacts in time and leads to the temperature jumps caused by the periodical rising of heat flow. The character of the temperature distribution is connected with the value of the only dimensionless parameter $B=a/\omega R^2=a/uR$ binding together a thermal property of used material, a characteristic size of a ring and a frequency of δ-source rotation. One can treat parameter B as a dimensionless coefficient of heat diffusivity.

3. Results and discussions

Time dependencies of normalized temperature $v=c\rho\Delta r\omega T/q$ in the point $\varphi=0$ for different values of B are shown in Fig. 1. At high values of the dimensionless thermal diffusivity $B\gg1$ temperature curves tend to have the form of a straight line. In this particular case heat conduction dominates in the system and one can say about the heating of a moving (without any rotation)

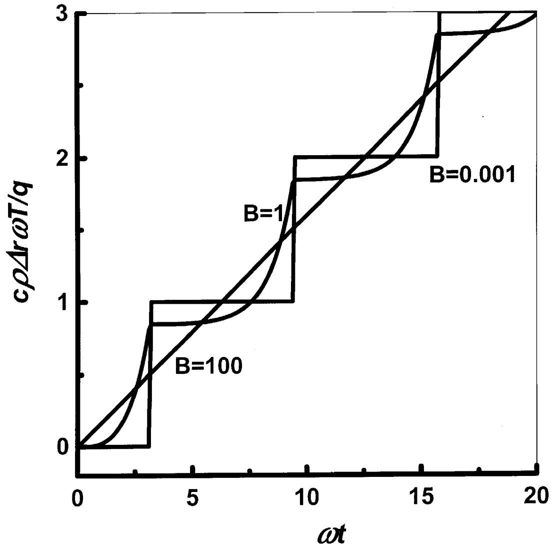

Fig. 1. Time dependencies of dimensionless temperature in the ring point $\varphi = 0$ for various values of dimensionless thermal diffusivity B.

material point in a "hot" medium. At low values of $B \ll 1$ temperature curves have characteristic stepped form and with each contact dimensionless temperature rises by 1. It means that heat conduction is negligible and temperature variations are only connected with direct contacts with a periodical δ-source of energy. Mathematically the form of temperature curve will be determined for all moments of time excepting moments of direct contacts with δ-source by the whole number term

$$\frac{c\rho\Delta r\omega T}{q} \approx \frac{\varphi - B + 2\pi}{2\pi} + \left[\frac{\tau - \varphi - \pi}{2\pi}\right] \tag{10}$$

that can be approximated with a high accuracy by means of a strict succession of positive whole numbers. In this case any device with real (limited) detectivity and time resolution will measure only discrete levels of temperature. The smaller size and higher accuracy the device has, the closer data measured to discrete levels are. Measuring the temperature we can find the number of ring revolutions and on the contrary the known number of revolutions allows us to find out the temperature with high accuracy.

At intermediate values of $B \sim 1$ all terms in (9) are of the same order and temperature curves consist of a succession of jointed "periodical" exponents:

$$\frac{c\rho\Delta r\omega T}{q} \approx \frac{\varphi - B + 2\pi}{2\pi} + \left[\frac{\tau - \varphi - \pi}{2\pi}\right] + \frac{\exp\left(-\frac{2\pi}{B}\left(1 - \left\{\frac{\tau-\varphi-\pi}{2\pi}\right\}\right)\right)}{1 - \exp(-2\pi/B)}. \tag{11}$$

The analysis of situation under consideration does not take into account thermal losses into surrounding medium. These losses (into medium under zero temperature) can be accounted in the mathematical model by adding the term $\alpha T/c\rho\Delta r$ to the right-hand side of Eq. (1). In this case the problem (1)–(4) will have the following exact solution:

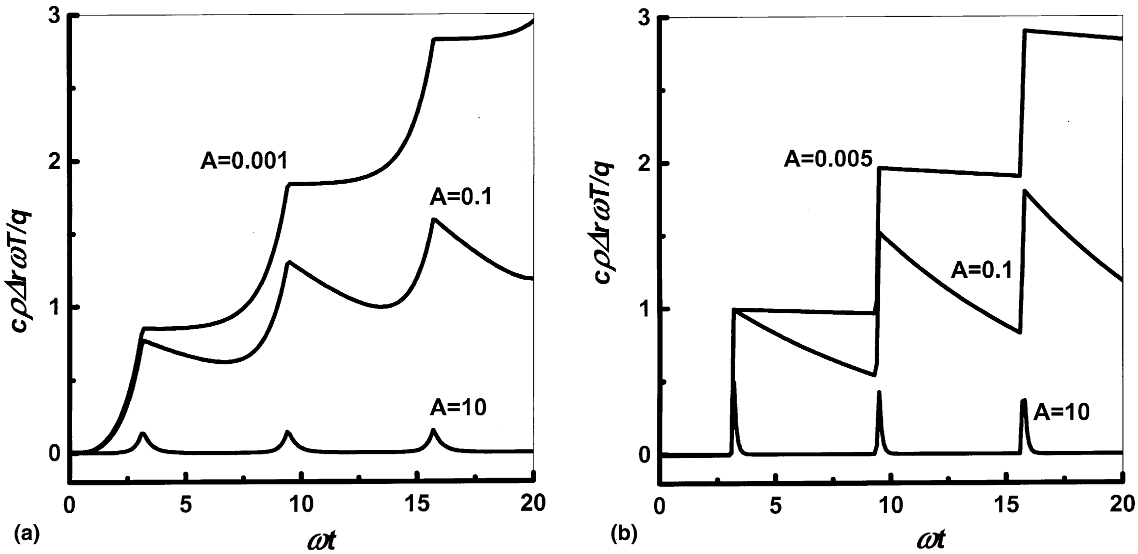

Fig. 2. Time dependencies of dimensionless temperature in the ring point $\varphi = 0$ for various values of dimensionless heat transfer coefficient A ((a) thermal diffusivity $B = 1$, (b) $B = 0.001$).

$$v(\varphi,\tau) = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k \exp(-(Bk^2+A)\tau)}{(Bk^2+A)+k^2} \times (k\sin(k\varphi) - (Bk^2+A)\cos(k\varphi)) - \frac{\exp(-A\tau)}{2\pi A} + \frac{1}{D} \times \frac{\exp\left(-2\pi D_+\left(1-\left\{\frac{\tau-\varphi-\pi}{2\pi}\right\}\right)\right)}{1-\exp(-2\pi D_+)} - \frac{1}{D} \times \frac{\exp\left(-2\pi D_-\left(1-\left\{\frac{\tau-\varphi-\pi}{2\pi}\right\}\right)\right)}{1-\exp(-2\pi D_-)}, \quad (12)$$

where $A = \alpha/c\rho\Delta r\omega$, $D = (4AB+1)^{1/2}$, $D_+ = (1+D)/2B$, $D_- = (1-D)/2B$.

At $\omega t \to \infty$ first and second terms will be negligibly small and temperature variation in a fixed point of a ring will be determined by the sum of two exponents with periodical fractional indices. Time dependencies of normalized temperature $v = c\rho\Delta r\omega T/q$ in the point $\varphi = 0$ for different values of the dimensionless heat transfer coefficient A and thermal diffusivity B are shown in Fig. 2. At low A the temperature curves approach the curves built in Fig. 1 for corresponding values of B. The rise of the heat transfer coefficient ($A = 0.1$) leads to the total lowering of the temperature level. The intensive heat transfer with medium (high values of $A \gg 1$) practically sets the temperature of surrounding medium within a ring excepting the time moments of the direct contacts with δ-source of energy. As one can see temperature curves in Fig. 2(b) ($B = 0.001$) practically compose of straightline sections, so, in spite of the cooling to medium, these curves have more "integer" character then ones in Fig. 2(a). The smaller the heat transfer coefficient is, the more integer curves are.

Thus, in the note new continuous stepped solutions of the well-known problem of heat conduction in solids, Fourier's problem of a ring heated by the moving concentrated source of energy, are obtained. These integer and fractional solutions are not mathematically the special case of any other known type of exact solutions of the linear heat conduction equation and can be used for the quantitative description of heat transfer processes in different systems, for example the moving caterpillar track. As a rule, parabolic equations in the theory of heat transfer describe dissipative processes that led to the flattening of initial or boundary discontinuities. The problem discussed above demonstrates the uncommon opposite situation – formation of abrupt stepped distributions of temperature at the straightforward and steady motion of rotating hollow objects even at a constant speed and constant rate of energy transfer with medium.

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References

[1] J.P.J. Fourier, The Analytical Theory of Heat, Dover, New York, 1952, p. 466.
[2] H.S. Carslaw, J.C. Jaeger, Conduction of Heat in Solids, Clarendon Press, Oxford, 1959, p. 510.
[3] G. Doetsch, Anleitung zum praktischen gebrauch der Laplace transformation und der z-transformation, Oldenbourg, Munchen, 1967, p. 288.